Differentiating (4) with regard to n, we have

$$\frac{\partial \phi}{\partial n} = 2n^{-2}(y-1)(y+1)^{2/n} \log (y+1) - 2n^{-2}(y+1)y^{2/n} \log y$$
$$= 2n^{-2}(y+1)(y-1)y[\psi(y+1) - \psi(y)],$$

where $\psi(y) = (y-1)^{-1}y^{(2-n)/n} \log y$.

For $n = 1$, $\psi(y) = (y \log y)/(y-1)$. This is an increasing function of y for all values of $y > 1$, as may be proved by differentiating it and then differentiating the numerator of its derivative.

Therefore $\partial\phi/\partial n > 0$ for $n = 1$.

For $n = 2$, $\psi(y) = (\log y)/(y-1)$; which is found in the same way to be a decreasing function of y for all values of $y > 1$.

Therefore $\partial\phi/\partial n < 0$ for $n = 2$.

If we equate $\partial\phi/\partial n$ to zero, putting $\psi(y+1) = \psi(y)$, and take the logarithms of both sides, we shall obtain a linear equation in n, satisfied by only one value of n for any given value of y. This value of n comes between 1 and 2 and, as $\phi(y, 1) = \phi(y, 2) = 0$, it follows that $\phi(y, n)$ is positive when n lies between 1 and 2, and negative when $n > 2$, and therefore that f is an increasing function of x in the former case and a decreasing function of x in the latter case.

When $x = 1$, $f = 1 - 2^{(2-n)/n}$ and as f represents cos XOY, we have the following results:

(1) *For any given $n > 2$, and for a given angle XOY, there exists one and only one independent pair of points P, Q on OX and OY, respectively, such that $\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n$, provided the angle XOY satisfies*

$$\text{arc cosine } (1 - 2^{-(n-2)/n}) \leqq XOY \leqq 90°;$$

otherwise such a pair of points does not exist.

(2) *For n between 1 and 2, and for a given angle XOY there exists one and only one pair of points satisfying the condition, provided the angle XOY satisfies*

$$\text{arc cosine } (1 - 2^{(2-n)/n}) \geqq XOY \geqq 90°;$$

otherwise there is no pair of points to satisfy the condition.

(3) *In either case this range decreases as n approaches 2, condensing to a single angle 90°, but for 90° any pair of points is a solution and there is no limit to the number of independent solutions.*

We can also state now this theorem:

If two triangles have an angle of one equal to an angle of the other, and if in each triangle the n-th power of the side opposite is equal to the sum of the n-th powers of the other two sides, then, for n greater than 1 and different from 2, the triangles are similar.

IV. Note on Trigonometric Functions.

By Elijah Swift, University of Vermont.

Mr. R. S. Underwood's note in this Monthly, *1921*, 374 (see also *1922*, 255, 346), concerning the irrationality of certain trigonometric functions, leads me to supplement his statements in one particular.

We call the sine and cosine transcendental functions and one might assume that their values as given in the usual table were, for the most part, transcendental numbers. So far from that being the case, none of the values of the natural functions listed in the usual trigonometric table are transcendental.

The proof is immediate. The angles whose functions are listed are all commensurable with π, being given in degrees and fractions of degrees. But if $\theta = m\pi/n$, where m and n are integers, then $\sin n\theta = \sin m\pi = 0$. But there is a well-known formula giving $\sin n\theta$ as a rational algebraic function of $\sin \theta$, if n is odd, or of $\sin \theta$ and $\cos \theta$, if n be even. In the latter case the equation may be rationalized in $\sin \theta$, and in either case on substituting zero for $\sin n\theta$ we find that $\sin \theta$ satisfies a rational algebraic equation with rational coefficients and is consequently an algebraic number. The fact that the values of the other trigonometric functions of θ are algebraic follows from the formulas connecting them with the value of $\sin \theta$.[1]

V. Slope of a Curve in Polar Coördinates at the Pole.

By H. J. Ettlinger, University of Texas.

The pole is an exceptional point in polar coördinates. For example to every point in the plane, except the pole, there corresponds one pair of coördinates, (ρ, θ), $\rho > 0$ and $0 \leqq \theta < 2\pi$. Conversely to each pair of such numbers there corresponds one point in the plane. The pole, however, corresponds to the single coördinate $\rho = 0$ and conversely.

At the pole, this fact produces a singular situation with respect to the slope of a curve, $\rho = f(\theta)$ or $\theta = F(\rho)$, which passes through it. At an ordinary point, the slope of the tangent is given as $m = \tan \varphi$, where $\varphi = \theta + \psi$ and ψ is the angle formed by the tangent at P with the radius OP. By the usual method, we find

$$\tan \psi = \rho \frac{d\theta}{d\rho} \cdot \tag{1}$$

At the pole this equation yields $\psi = 0$ and hence

$$\varphi = \theta. \tag{2}$$

To give a meaning to θ at $\rho = 0$, we consider a point P on the curve whose coördinates are $(\Delta\rho, \bar{\theta})$. As $\Delta\rho$ approaches zero, if $\bar{\theta}$ approaches a limit θ_0, this is the direction of the tangent to the curve at the pole. This justifies equation (2). To find this value of θ_0, we solve $f(\theta) = 0$ or $\theta_0 = F(0)$, where $F(\rho)$ is supposed continuous for $\rho = 0$. If the equation of the curve is so defined that to each value of ρ there corresponds only one value of θ, the slope of the tangent at the pole is uniquely determined as

$$m = \tan \theta_0.$$

[1] On the other hand, any angle whose radian measure is a rational number does have transcendental numbers for the values of its trigonometric functions. See F. Klein, *Famous Problems of Elementary Geometry*, translation by Beman and Smith, Boston, 1897, p. 77.